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On characterization of primitive commutative association schemes by a nonsymmetric relation with small valency

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Abstract

Let (X, G) be a primitive commutative association scheme with a nonsymmetric relation of valency 4. Then, for each nonsymmetric relation g of valency 4, the graph (X, g) is uniquely determined. In particular, the cardinality of X is the cube of an odd prime.

Let (X, G) be an association scheme (see [4]). Then for each $g \in G$, (X, g) is a regular graph satisfying certain conditions. It seems an interesting problem to determine whether a given graph could be a relation of an association scheme. This problem has been studied before and will be one of the main topics in characterizing association schemes by certain intersection numbers. Until now, quite a few results for this problem under various assumptions or interests, for instance, when (X, G) is a P-polynomial association scheme or a translation association scheme (see [4] or [6]), or when we impose conditions on valencies or cardinality of X, are obtained by many researchers. The author is especially interested in association schemes with a nonsymmetric relation, namely aims to determine the directed graph (X, g) with $g \in G$ or the structure of (X, G) under some assumptions, for example, when (X, G) is primitive, commutative or when $|X|$ is a prime.

The following is an example of such association schemes.

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Example 0.1 Let F_p be a finite field of order p where p is an odd prime. We define some permutations on F_p^n as follows;

- Let S_n be the symmetric group of degree n. For all $(x_1, x_2, \ldots, x_n) \in F_p^n$ and $\sigma \in S_n$,

$$(x_1, x_2, \ldots, x_n)^\sigma := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}).$$

- For each $(x_1, x_2, \ldots, x_n) \in F_p^n$, we define $\tau \in GL(F_p^n)$ as follows;

$$(x_1, x_2, \ldots, x_n)^\tau := (-x_1, x_2 - x_1, \ldots, x_n - x_1).$$

- For all $u, v \in F_p^n$,

$$u^v = u + v.$$

Then the permutation group $\Gamma := < S_n, \tau, F_p^n >$ acts on F_p^n transitively, and Γ acts on $F_p^n \times F_p^n$ by $(u,v)^g := (u^g, v^g)$ where $(u,v) \in F_p^n \times F_p^n$ and $g \in \Gamma$. Let G be the set of orbits of Γ on $F_p^n \times F_p^n$. Then (F_p^3, G) is an association scheme (see [4, p.52]), and it can be verified that (F_p^n, G) is a primitive commutative association scheme with a nonsymmetric relation of valency $n+1$. Indeed, the orbit containing $((0,0,\ldots,0),(1,0,\ldots,0))$ is a nonsymmetric relation of valency $n+1$.

Let (X, G) be a primitive commutative association scheme. The following results are obtained until now.

1) If there exists a nonsymmetric relation $g \in G$ of valency 1 then (X, g) is a directed cycle of prime length.

2) There exists (X, G) with a nonsymmeteric relation of valency 2.

3) If there exists a nonsymmetric relation $g \in G$ of valency 3 then we have one of the following;

   (a) (X, g) is isomorphic to a relation of a cyclotomic scheme where $|X|$ is a prime.

   (b) There exists a relation $\tilde{g} \in G$ isomorphic to a relation of a cyclotomic scheme where $|X|$ is an odd prime squared.

Although it is trivial to prove 1), 2), it is rather difficult to prove 3) (see [10]).

The following is a main result of this presentation.

Theorem 0.2 *Let (X, G) be a primitive commutative association scheme with a nonsymmetric relation of valency 4, denoted by g. Then the graph (X, g) is isomorphic to (F_p^3, E) where $(x, y) \in E$ if and only if $y - x \in \{(1,0,0),(0,1,0),(0,0,1),(-1,-1,-1)\}$.*

We need certain intersection numbers in order to determine the graph (X, g). The following proposition holds.

Proposition 0.3 *For each $g \in G$ with $k_g = 4$ and $g^* \neq g$, there exist a, b, f, h, $m \in G$ such that*

(i) $g \bullet g = 2a + b$, $k_a = 6$, $k_b = 4$.

(ii) $g \bullet g^* = 4 \cdot 1_X + h$, $k_h = 12$.

(iii) $a^* \bullet a = 6 \cdot 1_X + 2h + f$, $k_f = 6$.

(iv) $a = a^*$, $g \bullet a = 3g^* + m$, $k_m = 12$.

(v) $ga \cap gb = \{m\}$.

Definition 0.4 Let $e \in G$ be such that $k_e = 4$ and $e \neq e^*$. A sequence $(x_0, x_1, \ldots, x_j)$ of elements of X is *a chain* if $(x_i, x_{i+1}) \in g$ for each i with $0 \leq i \leq j-1$, and $(x_i, x_{i+2}) \in b$ for each i with $0 \leq i \leq j-2$.

Remark 0.5 *If $(x_0, x_1, \ldots, x_i)$ is a chain then there exists a unique element $x_{i+1} \in X$ such that $(x_{i-1}, x_{i+1}) \in b$ and $(x_i, x_{i+1}) \in g$ by $p^g_{bg^*} = 1$, and $(x_0, x_1, \ldots, x_i, x_{i+1})$ is also a chain. Hence we obtain a chain of length i for any $i > 0$, and whence there exists a closed chain since $|X|$ is finite.*

Let $(x_0, x_1, \ldots, x_n = x_0)$ be a closed chain of the shortest length. We would like to show that n is a prime and identify $(x_0, x_1, \ldots, x_n = x_0)$ with F_p. Proposition 0.3 makes the above things posssible. Moreover, we can construct F_p^3 by repeating to connect a closed chain $(y_0, y_1, \ldots, y_n = y_0)$ to another closed chain $(y'_0, y'_1, \ldots, y'_n = y'_0)$ such that $(y_i, y'_i) \in g$ for all i with $0 \leq i \leq n-1$ and $y'_0 \neq y_1$.

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